we are not obliged to limit the required solutions only to the class of continuous functions. From both theoretical considerations and analysis of the experimental data, it is well known (see, for example, [2]), that in the region between the center of the explosion and the perturbation front, with defined conditions, strong discontinuities can originate. Under the conditions of the example chosen and within the framework of the approximations considered here, these discontinuities are not developed.

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REPRESENTATION OF INTERACTION IN THE THEORY OF TURBULENCE

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The concept that in a turbulent flow energy exchanges only take place of pulsations of near scales is the basis of macroscopic theory of local turbulence structure. Universality and similarity of small-scale statistical pulsations are inferred from the assumption that the energy exchange is of random character. In the Eulerian equations of motion, together with the interactions which implement the energy exchange between pulsations, there are fictitious interactions related to the transfer of pulsations of a given scale l by the pulsations of scales $l' \gg l$. It was emphasized in [2, 3] that in the Eulerian description of turbulence the effect of transfer results in a strong statistical dependence of pulsations of different scales. Therefore, the universality and similarity of small-scale pulsations can be observed only in these variables in which there are no effects of pure transfer of some pulsations by the others. Qualitative considerations were therefore given in [1-3] on the need for describing small-scale pulsations in a reference system which is in motion at each point with all large-scale pulsations. It is shown in the present article that such description of small-scale pulsations can be implemented with the aid of transfer representation similar to the representation of interaction in the quantum field theory [4]. Representation of interaction is of intermediate position between the Lagrangian and Eulerian descriptions of turbulence, since a transfer of a packet as an entity can be described in variables which are Lagrangian only as regards large-scale motions. Another way of eliminating transfer interactions is based on the introduction of nonsolenoid velocity as in [5]. From the physical point of view, the method employed in this article seems to be more appropriate.

First, the case of the scalar field $\varphi(\mathbf{x}, t)$ is considered; its entire evolution in time is related to the transfer of the field φ to the velocity field $\mathbf{v}(\mathbf{x}, t)$. The part of the field φ can be taken, for example, by the concentration of a passive admixture in a turbulent flow. The equation for φ is

$$\partial\varphi/\partial t + (\mathbf{v}\nabla)\varphi = 0. \tag{1}$$

By integrating (1) with respect to time one obtains the integral equation

$$\varphi(\mathbf{x}, t) = \varphi(\mathbf{x}, t_0) - \int_{t_0}^{t} d\tau\, (\mathbf{v}(\mathbf{x}, \tau)\nabla)\, \varphi(\mathbf{x}, \tau).$$

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The solution of this equation can be written in the form of an iterative series

$$\varphi(\mathbf{x}, t) = L\varphi(\mathbf{x}, t_0) \equiv \left[1 + \sum_{n=1}^{\infty} (-1)^n \int_{t_0}^{t} d\tau_1 (\mathbf{v}(\mathbf{x}, \tau_1)\nabla) \int_{t_0}^{\tau_1} d\tau_2 \ldots \ldots \int_{t_0}^{\tau_{n-1}} d\tau_n (\mathbf{v}(\mathbf{x}, \tau_n)\nabla)\right] \varphi(\mathbf{x}, t_0). \tag{2}$$

The integration with respect to all $d\tau_m$ can be extended to the entire interval (t_0, t) by introducing the T-ordering operation [4]. By definition one has

$$T[(\mathbf{v}(\mathbf{x}, \tau)\nabla)(\mathbf{v}(\mathbf{x}, \tau')\nabla) = \begin{cases}(\mathbf{v}(\mathbf{x}, \tau)\nabla)(\mathbf{v}(\mathbf{x}, \tau')\nabla), & \text{if} \quad \tau > \tau', \\ (\mathbf{v}(\mathbf{x}, \tau')\nabla)(\mathbf{v}(\mathbf{x}, \tau)\nabla), & \text{if} \quad \tau' > \tau.\end{cases}$$

In the case of a large number of multiplicants the T-ordering operator arranges the noncommuting operators ($\mathbf{v}\nabla$) in decreasing order of the time arguments from left to right. The operator L in the formula (2) can be written with the aid of the T-ordering operator as [4]

$$L = \left\{1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{t_0}^{t} d\tau_1 \ldots \int_{t_0}^{t} d\tau_n T[(\mathbf{v}(\mathbf{x}, \tau_1)\nabla) \ldots (\mathbf{v}(\mathbf{x}, \tau_n)\nabla)]\right\} = T\exp\left[-\int_{t_0}^{t} d\tau (\mathbf{v}(\mathbf{x}, \tau)\ \nabla)\right]. \tag{3}$$

In accordance with (2) the operator L links the value of the function $\varphi(\mathbf{x}, t)$ at any time instant with its value at a constant time instant t_0. In this sense the operator L implements for Eq. (1) the transition to a representation similar to that of Heisenberg in quantum mechanics. If the field φ is transformed according to the formula $\varphi(\mathbf{x}, t) = L\tilde{\varphi}(\mathbf{x}, t)$, then the field $\tilde{\varphi}$ becomes independent of time being equal to the value of the field φ at the instant t_0. In the fluid flow the concentration of the admixture remains constant along the Lagrangian trajectories of the particles. The operator L can, therefore, also be understood in another sense. For any field $\psi(\mathbf{x}, t)$ (which need not be a scalar one) the operator L describes the transition to the Lagrangian variables. The field $\tilde{\psi}$ specified by

$$\psi(\mathbf{x}, t) = L\tilde{\psi}(\mathbf{x}, t), \tag{4}$$

is a Lagrangian field. The assertion can also be proved in a different way. The field $\tilde{\psi}$ is expanded in the Taylor series in $t - t_0$,

$$\tilde{\psi}(\mathbf{x}, t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \frac{\partial^n \tilde{\psi}}{\partial t^n}\Big|_{t=t_0}. \tag{5}$$

By using (4) the series (5) can be rewritten as

$$\tilde{\psi}(\mathbf{x}, t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \left(\frac{\partial}{\partial t} + \mathbf{v}\nabla\right)^n \psi(\mathbf{x}, t)\Big|_{t=t_0}. \tag{6}$$

According to [6] the expression (6) is the Eulerian link between the field ψ and the Lagrangian $\tilde{\psi}$.

The point x in (4) is the Eulerian coordinate for the field ψ and the Lagrangian one for the field $\tilde{\psi}$. Usually, the relation between the Lagrangian and the Eulerian fields is given by

$$\psi(\mathbf{x}, t) = \tilde{\psi}(\mathbf{a}, t),$$

where $\mathbf{a} = \mathbf{x} - \int_{t_0}^{t} \tilde{\mathbf{v}}(\mathbf{a}, \tau)\, d\tau$, $\tilde{\mathbf{v}}$ is the Lagrangian velocity. Consequently, the field ψ on the left is equal to the Lagrangian field $\tilde{\psi}$ at the point shifted from the original one by $\int_{t_0}^{t} \tilde{\mathbf{v}} d\tau$. Hence, one can obtain the properties of the operator L which are required below. If $\tilde{\psi}$, $\tilde{\chi}$ are arbitrary Lagrangian fields, then

$$L\ (\tilde{\psi}\ \tilde{\chi}) = L(\tilde{\psi})\ L(\tilde{\chi}), \quad L(\tilde{\psi})^n = (L\tilde{\psi})^n. \tag{7}$$

The relations (7) could also be proved directly from the definition of the operator L.

The operator L^{-1} inverse to L is now found. To this end one used the equation satisfied by any Lagrangian field $\tilde{\psi}(\mathbf{x}, t, t_0)$ [5],

$$[\partial/\partial t_0 + \mathbf{v}(\mathbf{x}, t_0)\nabla]\tilde{\psi}(\mathbf{x}, t, t_0) = 0. \tag{8}$$

Equation (8) results from the fact that the value of a Lagrangian field measured at the time instant t does not change if the initial values $\mathbf{x}$, t_0 are shifted along the Lagrangian trajectory. For $t = t_0$ the field $\tilde{\psi}$ is identical with the Eulerian field ψ; therefore, the solution of Eq. (8) can be written in a form similar to (2),

$$\tilde{\psi}(\mathbf{x}, t, t_0) = \left[1 + \sum_{n=1}^{\infty} (-1)^n \int_t^{t_0} d\tau_1 (\mathbf{v}(\mathbf{x}, \tau_1)\nabla) \int_t^{\tau_1} d\tau_2 \ldots \right.$$
$$\left. \ldots \int_t^{\tau_{n-1}} d\tau_n (\mathbf{v}(\mathbf{x}, \tau_n)\nabla)\right] \psi(\mathbf{x}, t).$$

The inversion of the integration limits produces

$$\tilde{\psi}(\mathbf{x}, t, t_0) = \left[1 + \sum_{n=1}^{\infty} \int_{t_0}^{t} d\tau_1 (\mathbf{v}\nabla) \int_{\tau_1}^{t} d\tau_2 \ldots \int_{\tau_{n-1}}^{t} d\tau_n (\mathbf{v}\nabla)\right] \psi(\mathbf{x}, t).$$

The operation of T^{+} ordering is now introduced. The operator T^{+} arranges the operators $(\mathbf{v}\nabla)$ in increasing order of time arguments from left to right. The sum of the series can be written as

$$\tilde{\psi}(\mathbf{x}, t, t_0) = L^{-1}\psi(\mathbf{x}, t) = T^{+} \exp\left[\int_{t_0}^{t} d\tau \mathbf{v}(\mathbf{x}, \tau)\nabla\right] \psi(\mathbf{x}, t). \tag{9}$$

It should be observed that the sign in the index of the exponential is inverse to the location order of the operators in the terms of the expansion in the relation (2) and (9).

Let the field ψ satisfy a more general equation than (1), namely,

$$\partial\psi/\partial t + (\mathbf{v}\nabla)\psi = P(\psi, \nabla\psi), \tag{10}$$

where P is a polynomial in $\psi(\mathbf{x}, t)$ and in its space derivatives. An equation is now considered which is satisfied by the Lagrangian field $\tilde{\psi}(\mathbf{x}, t, t_0)$. By inserting (4) into (10) one obtains

$$\partial\tilde{\varphi}/\partial t = L^{-1}P(L\tilde{\psi}, \nabla L\tilde{\psi}). \tag{11}$$

By virtue of the relation (7), Eq. (11) assumes the form

$$\partial\tilde{\psi}/\partial t = P(\tilde{\psi}, L^{-1}\nabla L\tilde{\psi}). \tag{12}$$

Hence, it follows that if the commutator of the derivative ∇ and the operator L can be ignored, then the equation for $\tilde{\psi}$ is reduced to

$$\partial\tilde{\psi}/\partial t = P(\tilde{\psi}, \nabla\tilde{\psi}), \tag{13}$$

that is, in this case Eq. (13) is equivalent to (10) with $\mathbf{v} = 0$. The commutators of the operator L and the derivatives are of the order of the quantity $(t - t_0)\partial v_i/\partial x_j$. Therefore, the solutions of (12) are identical with those of Eq. (13) provided that the inequality

$$|t - t_0| \ll |\text{grad } \mathbf{v}|^{-1}, \tag{14}$$

is valid where $|\text{grad } \mathbf{v}|^{-1}$ is on the order of magnitude of time in which two near points have had time to be separated by a considerable distance. This, however, does not imply that if (14) holds, then the statistical state of the field $\tilde{\psi}$ is independent of $\mathbf{v}$ because, in general, the initial conditions for the field $\tilde{\psi}$ depend on $\mathbf{v}$. At the initial time instant the Lagrangian field $\tilde{\psi}$ is identical with the Eulerian field ψ. Therefore, the statistical state of the initial conditions of $\tilde{\psi}(\mathbf{x}, t, t_0)$ can only be determined by the simultaneous moments of the Eulerian field $\psi(\mathbf{x}, t_0)$.

It will be shown that the simultaneous moments of the Eulerian field ψ are independent of $\mathbf{v}$ if the space and time scales of the field $\mathbf{v}$ are large compared with the characteristic scale of the field ψ. First, a simple example is considered with the field $\mathbf{v}$ independent of $\mathbf{x}$, t being a random vector with known statistical properties. In this case the transformation (4) describes a transition to another Galileo reference system which is in motion together with the fluid with velocity **v**,

$$\psi(\mathbf{x}, t) = \exp[-(t - t_0)(\mathbf{v}\nabla)]\tilde{\psi}(\mathbf{x}, t) = \tilde{\psi}(\mathbf{x} - \mathbf{v}(t - t_0), t). \tag{15}$$

The averages of the Lagrangian field $\tilde{\psi}$,

$$\tilde{G}_n = \langle\tilde{\psi}(\mathbf{x}_1, t_1) \ldots \tilde{\psi}(\mathbf{x}_n, t_n)\rangle \tag{16}$$

are identical in this case with the averages of the field ψ° in the fluid at rest and are independent of **v**. The moments G_n of the Eulerian field ψ can be determined with the aid of (15) by averaging over the probability distribution of the vector **v**. Let the field $\tilde{\psi}$ be statistically uniform. The simplest relation between the functions G_n, $\tilde{G}_n$ is by means of a Fourier representation in the space arguments. The transform (15) for the Fourier components of $\psi(\mathbf{k}, t)$ is

$$\psi(\mathbf{k}, t) = \exp[-i\mathbf{k}\mathbf{v}(t - t_0)]\tilde{\psi}(\mathbf{k}, t).$$

Hence, one obtains

$$G_n(\mathbf{k}_1, t_1, \ldots, \mathbf{k}_n, t_n) = Z(\boldsymbol{\alpha})\tilde{G}_n(\mathbf{k}_1, t_1, \ldots, \mathbf{k}_n, t_n), \tag{17}$$

where $\boldsymbol{\alpha} = \sum_{i=1}^{n} \mathbf{k}_i (t_i - t_0)$; $Z(\boldsymbol{\alpha}) = \langle \exp(-i\mathbf{v}\boldsymbol{\alpha})\rangle$ is the characteristic function of the random vector **v**. For example, for a Gaussian **v** its characteristic function Z_0 is given by [7]

$$Z_0(\alpha) = \exp[-i\langle v_j\rangle \alpha_j - (1/2)\langle v_j v_m\rangle \alpha_j\alpha_m].$$

Moreover, if the probability distribution for **v** is isotropic, then $\langle \mathbf{v}\rangle = 0$.

$$\langle v_j v_m\rangle = v_0^2\delta_{jm} \text{ and } Z_0(\boldsymbol{\alpha}) = \exp\left(-\frac{1}{2}v_0^2\alpha^2\right).$$

In view of spatial uniformity one has $\sum_{i=1}^{n} \mathbf{k}_i = 0$. Therefore, $\boldsymbol{\alpha} = \sum_{i=1}^{n-1} \mathbf{k}_i\tau_i$, where $\tau_i = t_i - t_n$. Hence, it follows that simultaneous moments of all orders of an Eulerian field ψ are identical with the averages of the field $\tilde{\psi}$ and are independent of **v**. For single-time averages one has $\alpha = 0$, $Z(0) = 1$, $G_n = \tilde{G}_n$.

In the derivation of (17) only the assumption on the statistical independence of the field $\tilde{\psi}$ was used, which is identical with the field ψ° in the fluid at rest from the velocity field **v**. Therefore, the result in (17) does not depend on the actual form of the equation which describes the evolution of the field ψ. The formula (17) for a vector field **u** which satisfies the transport equation with **v** = const

$$\partial\mathbf{u}/\partial t + (\mathbf{v}\nabla)\mathbf{u} = 0,$$

in the case of n = 2 and under the assumptions that the distribution is normal and the fields **v**, $\mathbf{u}(\mathbf{x}, t_0)$ are statistically independent was obtained in [3].

Now let the field **v** depend on **x**, t. In a fluid turbulent flow as shown in [1] it can be expected that different in-scale motions are statistically independent. The same considerations, though we cannot claim to possess a rigorous proof, can be employed in our case. Let the correlation time τ_c of a stationary random field $\psi^{\circ}(\mathbf{x}, t)$ in a fluid at rest be insignificant compared with the time in which the velocity **v** changes, and the correlation length r_c of the field ψ° be small compared with the scales L of the velocity field $\mathbf{v}(\mathbf{x}, t)$. The interaction between the fields **v**, ψ can, in the main, be reduced to the transfer with no noticeable distortion of the wave packets of the field ψ and the field **v**. The evolution of the packets in a reference system which is in motion with the fluid is determined mainly by the nonlinear interaction P. In a reference system in motion the interaction between the field packets ψ and the field **v** is small in the parameters r_c/L and $\tau_c L/v$. Therefore, the statistical dependence of the packets of the field ψ on **v** is also weak. The simultaneous correlations of the field ψ which do not depend on transport with an accuracy up to small terms in r_c/L, $\tau L/v$ are independent of **v**.

Since Eq. (13) and the initial values for $\tilde{\psi}$ do not in the limit $r_c/L \to 0$ depend on **v**, the nonsimultaneous correlation functions for the field ψ under the condition (14) are independent of **v** and are equal to the correlation functions for the field ψ° in the fluid at rest. Thus, in the case under consideration the transformation (4) describes the transition to the "representation of interaction": It eliminates the fictitious part of the interaction related to the pure transport of the wave packets of the field ψ. If **v** contains harmonics of the same scales as ψ, then in the transition to the representation of interaction only the large-scale component of the velocity **v** should be retained in the index of the T exponent in the formula (4). In this case the equation for the field $\tilde{\psi}$ contains the interaction only with this component of the field **v**, which brings about the distortion of the packets for the field ψ. A transformation of this kind is employed below in analyzing the statistical state of velocity pulsations in the inertia gap interval of the wave numbers.

The nonsimultaneous elements of the field ψ are now evaluated taking into account that the correlations of the field ψ in a liquid at rest are given. If all the differences between the times $t_{ij} = |t_i - t_j|$ and the distance $x_{ij} = |\mathbf{x}_i - \mathbf{x}_j|$ in (16) are small compared with the characteristic scales of the field **v**, then in evaluating the functions G_n one can disregard the dependence of the field $\mathbf{v}(\mathbf{x}, t)$ on the coordinates and on time, and one can treat the field **v** as a random vector. This case was analyzed above. The function $Z(\boldsymbol{\alpha})$ in (17) should, in this case, be regarded as a single-point characteristic function of the field **v**. The simultaneous moments G_n, $\tilde{G}_n$ are equal. The nonsingular moments differ only slightly if the time differences t_{ij} are small compared with $(kv_0)^{-1}$, where v_0 is the mean-square pulsation of the vector **v**. The quantity $(kv_0)^{-1}$ is of the order of magnitude of the time needed for the displacement of the wave packet with the characteristic wave number k by a distance of the order of k^{-1}. If the correlations of the field ψ decay in a time much shorter than $(kv_0)^{-1}$, then the multiplier $Z(\boldsymbol{\alpha})$ in (16) can be disregarded. Of course, the transfer interactions are not essential for velocity pulsations in the viscous interval for asymptotically high wave numbers. In the interval of viscous dissipation of energy the characteristic decay time is $\tau_k \sim (\nu k^2)^{-1}$. For sufficiently high k the decay of the wave packets takes place earlier than their noticeable displacement. Therefore, for $k \gg v_0/\nu$ the transfer does not exert any effect on the dependence of the correlation functions of the velocity field on time. If our assumption of a sufficiently rapid decay of the wave packets is not satisfied for a field then the dependence of the Eulerian correlations on time can be fully determined for suitable high v_0 by the transport process.

Let us suppose that the points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in (16) can be subdivided into two groups so that the distances between the points x_{ij} within a group are small compared with the distance R between the groups of points. The latter quantity is regarded as large if one compares it with the correlation radius of the field $\tilde{\psi}$ and it is of comparable size with the characteristic scale of the field **v**. In this case, within each group of points one can introduce the general velocity $\mathbf{v}_1$, $\mathbf{v}_2$. The averages of the field $\tilde{\psi}$ can be split into a product of average fields belonging to each of the group of points:

$$\langle\tilde{\psi}(\mathbf{x}_1, t_1)\ldots\tilde{\psi}(\mathbf{x}_n, t_n)\rangle = \langle\tilde{\psi}(\mathbf{x}_1, t_1)\ldots\tilde{\psi}(\mathbf{x}_l, t_l)\rangle \times \langle\tilde{\psi}(\mathbf{x}'_1, t'_1)\ldots\tilde{\psi}(\mathbf{x}'_m, t'_m)\rangle, \quad (18)$$

where $l + m = n$. By applying to every average on the right of (18) the transform (15) with $\mathbf{v}_1$, $\mathbf{v}_2$, respectively, and by going over to the Fourier components as in (17) one obtains the formula

$$G_n = Z(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)\,\tilde{G}_n, \quad (19)$$

where $\boldsymbol{\alpha}_1 = \sum_{i=1}^{l} \mathbf{k}_i t_i$; $\boldsymbol{\alpha}_2 = \sum_{j=1}^{m} \mathbf{k}'_j t'_j$; $Z(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$ is the two-point characteristic function of the field **v**.

The sum of the wave vectors vanishes within each group of points. Therefore, $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ only depend on the time differences, the simultaneous moments G_n, $\tilde{G}_n$ being equal. A generalization of (19) to a greater number of points is obvious.

The obtained results are now employed to study the time correlations of the velocity field in the inertia interval of the wave numbers. The similarity considerations [1, 8] enable one to assume that the lifetime of the wave packets within the inertia interval are as powers of the wave numbers, $\tau_k \sim k^{-\varkappa}$. If it is assumed that the mean velocity ε of the energy dissipation is the only determining parameter of the pulsation state in the inertia interval, then by using dimensional concepts (see [1]) one obtains

$$\tau_k \sim \varepsilon^{-1/3} k^{-2/3}.$$

This time is considerable compared with the value $(kv_0)^{-1}$, where by v_0 one understands the characteristic pulsation rate from the energy containing interval. Therefore, the time dependence of the correlation functions for the Eulerian velocity in the inertia interval is fully determined by transfer random motions of a large scale. The latter indicates that the transfer interactions result in a strong statistical dependence of pulsations of Eulerian velocity of substantially different scales (see [3]). It is shown that mutual transfers of vortices with considerably different scales can be eliminated by interaction representation, which for the field ψ is similar to (4).

Let $\mathbf{v}(\mathbf{x}, t)$ be an Eulerian velocity field. The Lagrangian field $\tilde{\mathbf{v}}$ is related to $\mathbf{v}$ by an expression similar to (4),

$$\mathbf{v}(\mathbf{x}, t) = T \exp\left[- \int_{t_0}^{t} d\tau \mathbf{v}(\mathbf{x}, \tau) \nabla \right] \tilde{\mathbf{v}}(\mathbf{x}, t). \tag{20}$$

The inverse transformation is given by the T^{+}-exponent operator. It should be mentioned that the transition to the Lagrangian variables excludes the transfer of a material particle by motions of any scales. In our considerations the part of such a particle is taken by the wave packet. For motion of the scales $l' \gg l$ where l is the packet size, the wave packet can be regarded as a material point and one can, as regards these motions, proceed to the Lagrangian variables. In implementing this program in the coordinate space one has to expand the Eulerian velocity field into a complete system of functions of a wave packet kind and apply the transform (20) to each packet separately where in the index of the T exponent only the large-scale compared with the size of the packet component of the velocity field should be left. In eliminating pure transfer interactions the form of the wave packets is immaterial.

However, it is easier to start from the Navier-Stokes equation in its Fourier transform in the space variables,

$$(\partial/\partial t + \nu k^2) v_i(\mathbf{k}, t) = - i k_j \int d^3 q v_j(\mathbf{q}, t) v_i(\mathbf{k} - \mathbf{q}, t) + i k_i p(\mathbf{k}, t). \tag{21}$$

The incompressibility equation for the field $\mathbf{v}$ given by

$$\mathbf{k}\mathbf{v}(\mathbf{k}, t) = 0$$

enables one to express the Fourier component of the pressure field in terms of the velocity by means of

$$p(\mathbf{k}, t) = (k_j k_l / k^2) \int d^3 q v_j(\mathbf{q}, t) v_l(\mathbf{k} - \mathbf{q}, t). \tag{22}$$

The velocity $\mathbf{V}^{(k)}(\varkappa, t)$ is now introduced; it contains the Fourier harmonics $\mathbf{v}(\varkappa, t)$ with the wave numbers $\varkappa$ which are much smaller than k. For example, $\mathbf{V}$ is understood to be given by

$$\mathbf{V}^{(k)}(\varkappa, t) = \exp(-\lambda^2 \varkappa^2 / k^2) \mathbf{v}(\varkappa, t),$$

where $\lambda \gg 1$. In coordinate representation $\mathbf{V}$ is given by

$$\mathbf{V}^{(k)}(\mathbf{x}, t) = [k/(2\lambda \sqrt{\pi})]^3 \int_{-\infty}^{\infty} d^3 x' \exp[-(k^2/4\lambda^2)(\mathbf{x} - \mathbf{x}')^2] \mathbf{v}(\mathbf{x}', t), \tag{23}$$

That is, $\mathbf{V}$ is the velocity which smoothed over a volume much larger than k^{-1}. The contribution to the integrals (21) and (22) is now considered from the region of small $|\mathbf{q}|$, $|\mathbf{k} - \mathbf{q}|$. Since the multiplier $k_j k_l$ is present the contribution to the integral (22) from the region of small $|\mathbf{q}|$, $|\mathbf{k} - \mathbf{q}|$ is proportional to the gradient of the large-scale component of the velocity $\mathbf{V}$ (23); therefore, the contribution is small with respect to the parameter λ^{-1}. The contribution to the integral (21) from the region of small $|\mathbf{k} - \mathbf{q}|$ is also proportional to the gradient $\mathbf{V}$. The contribution to the integral (21) from the region of $|\mathbf{q}| \ll |\mathbf{k}|$ is of the order of magnitude of the velocity $\mathbf{V}$ and it is therefore also large. It was mentioned in [2] that this contribution describes a pure pulsation transfer of the scale k^{-1} by high-scale pulsations.

The transition to the representation of interaction for the Fourier components of the velocity field can be described with the aid of the transform

$$v_i(\mathbf{k}, t) = T \exp\left[- i k_j \int_{t_0}^{t} H_j(\tau) d\tau \right] \tilde{v}_i(\mathbf{k}, t), \tag{24}$$

where

$$H_j(\tau) = \int d^3\varkappa V_j^{(h)}(\varkappa, \tau) \exp(-\varkappa\partial/\partial\mathbf{k}). \quad (25)$$

To confirm the above, our considerations which resulted in (3) should be repeated for the transfer equation (1) in its Fourier representation. It will be shown that the equation for the field $\tilde{\mathbf{v}}$ determined by (24) contains no interactions related to mutual transfers of vertices whose scale differs by more than λ times. Since the space gradient of the large-scale component of **V** is small, in a low approximation in λ^{-1} the operator $\exp(-\varkappa\partial/\partial\mathbf{k})$ in (25) can be replaced by unity. The transform (24) then assumes the form

$$v_i(\mathbf{k}, t) = \exp\left[-ik_j \int_{t_0}^{t} d\tau \int d^3\varkappa V_j^k(\varkappa, \tau)\right] \tilde{v}_i(\mathbf{k}, t). \quad (26)$$

In this approximation the field $\tilde{\mathbf{v}}$ satisfies the incompressibility equation,

$$\mathbf{k}\tilde{\mathbf{v}}(\mathbf{k}, t) = 0. \quad (27)$$

By inserting (26) into (21) one has

$$(\partial/\partial t + \nu k^2)\tilde{v}_i(\mathbf{k}, t) = -ik_j \int d^3q\,[1 - \exp(-\lambda^2 q^2/k^2)] \times$$
$$\times \tilde{v}_j(\mathbf{q}, t)\tilde{v}_i(\mathbf{k}-\mathbf{q}, t) \exp\left\{i \int_{t_0}^{t} d\tau \int d^3\varkappa \left[\mathbf{q}\left(\mathbf{V}^{|\mathbf{q}|}(\varkappa, \tau) - \mathbf{V}^{|\mathbf{k}-\mathbf{q}|}(\varkappa, \tau)\right) + \right.\right. \quad (28)$$
$$\left.\left. + \mathbf{k}\left(\mathbf{V}^{|\mathbf{k}-\mathbf{q}|}(\varkappa, \tau) - \mathbf{V}^{|\mathbf{k}|}(\varkappa, \tau)\right)\right]\right\} + ik_i\tilde{p}(\mathbf{k}, t).$$

It can be shown that for finite $|\mathbf{q}|$, $|\mathbf{k}-\mathbf{q}|$ (this ensures a rapid convergence of the integral with respect to d^3q in the region of small $|\mathbf{q}|$, $|\mathbf{k}-\mathbf{q}|$) the index of the exponential in (28) is small in the parameter λ^{-1}. Therefore, in a poor approximation in λ^{-1} and by taking into account (27) one obtains for $\tilde{\mathbf{v}}$ the equation

$$(\partial/\partial t + \nu k^2) v_i(\mathbf{k}, t) = -ik_j\Delta_{il} \int d^3q \left[1 - \exp\left(-\frac{\lambda^2 q^2}{k^2}\right)\right] v_j(\mathbf{q}, t) v_l(\mathbf{k}-\mathbf{q}, t), \quad (29)$$

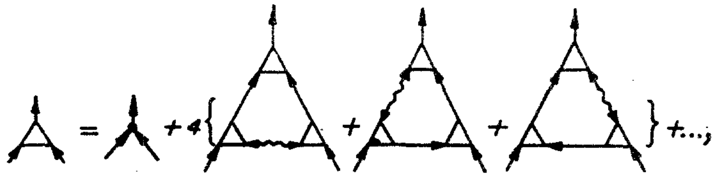
where

$$\Delta_{il} = \delta_{il} - k_i k_l/k^2.$$

If one takes into account the subsequent terms of the expansion in λ^{-1} this results in the appearance in the equation for $\tilde{\mathbf{v}}$ of nonlinearities of higher orders. The integral on the right of (29) is convergent in the region of $q \ll k$; therefore, there are no transfer interactions in this equation.

An equation similar to (29) was employed in [2] to obtain an improved approximation for direct interactions. The same equation can also be employed to obtain a complete system of diagram equations for the statistical characteristics of the field $\tilde{\mathbf{v}}$. To this end the Navier–Stokes (21) equation was employed directly in [8, 9] with a random external force. The assumption that the statistical characteristics of the Eulerian field **v** are similar is not inconsistent provided the prime apex has actually been shortened in the region in which the arguments of the lines belonging to it are appreciably different, that is, when the transfer interactions play an insignificant part. The transfer interactions can, for example, be ignored when small vortices arise and remain in between larger vortices. If this assumption is invalid, then the transfer interactions result in the integrals diverging in the region of low wave numbers. In this case, to study the similarity features one has to make use of Eq. (29) for semi-Lagrangian velocity $\tilde{\mathbf{v}}$. The prime apex of Eq. (29) diminishes rapidly outside the region in which its arguments are of the same order of magnitude and therefore no difficulties arise due to divergence. The problem now assumes the form similar to the problems in the phase-transition theory formulated in [10].

The general pattern of our considerations is now described to show how the similarity properties appear in the exact equations of the theory. The diagram technique for Eq. (29) and its analysis are similar to those in [8]. One adds to the right-hand side of Eq. (29) a random force with a nonvanishing spectrum only in the region of low wave numbers; one then proceeds to its Fourier representation with respect to time. By substituting in the mean field $\tilde{\mathbf{v}}$ the series-functions expansion in the external force and by adding partially the obtained series one arrives at the complete system of diagram equations for the spectral tensor F, the Green's tensor G, and for the vortex functions. Let us consider, for example, the equation for the vortex functions Γ [8, 9]



(30)

in which the following notation has been introduced:

$$\Delta \Leftrightarrow \Gamma_{ijl}(\mathbf{k}, \omega, \mathbf{q}, \Omega),$$

$$\leftarrow \Leftrightarrow G_{ij}(\mathbf{k}, \omega),$$

$$\dashleftarrow \Leftrightarrow F_{ij}(\mathbf{k}, \omega),$$

$$\Leftrightarrow P_{ijl}(\mathbf{k}) = -(i/2)(k_j\Delta_{il} + k_l\Delta_{ij}).$$

It is assumed that the tensors F, G, Γ are homogeneous functions of their arguments of degrees δ, β, —γ, respectively,

$$F_{ij}(\mathbf{k}, \omega) = k^{-\delta}F'_{ij}\left(\frac{\omega}{k^{\alpha}}\right), \; G_{ij}(\mathbf{k}, \omega) = k^{-\beta}G'_{ij}\left(\frac{\omega}{k^{\alpha}}\right),$$
$$\Gamma_{ijl}(\mathbf{k}, \omega, \mathbf{q}, \Omega) = k^{\gamma}\Gamma'_{ijl}\left(\frac{\omega}{k^{\alpha}}, \frac{\Omega}{q^{\alpha}}, \frac{\mathbf{q}}{k}\right). \tag{31}$$

The actual parameter in the expansion of the series (30) is $\mu \sim FG^2\Gamma^2k^3\omega$. Substituting (31) in any term of the series (30) and assuming that the main contribution to the integrals comes from the region in which the integration variables are on the order of magnitude of the outer lines one finds that all terms of the series are of the same degree of homogeneity as the left-hand side provided that $\mu \sim$ const ~ 1. This condition gives the following relation between the indices α, β, γ, δ:

$$2\gamma - 2\beta - \delta + \alpha + 3 = 0.$$

The formulas in [1] are satisfied for $\gamma = 1$, $\alpha = \beta = 2/3$, $\delta = 13/3$. The case of $\gamma \neq 1$ is consistent with Eq. (30) if, for example, the full vortex is large compared with the hunted one. Similar analysis holds also for other equations of the system.

The part played by the parameter λ is now discussed in some detail. For $\lambda = \infty$ one has $v^{(k)}(\varkappa, t) = 0$ and the field $\tilde{\mathbf{v}}$ is identical with the Eulerian velocity **v**. For $\lambda = 0$ the field **v** is the Lagrangian velocity. There is no applicability region for Eq. (29) in this case. One can ignore the higher-order nonlinearities in the equation for $\tilde{\mathbf{v}}$ if λ is sufficiently large. It is assumed that this can be done for $\lambda > \lambda_0$ where $\lambda_0 \gg 1$ is a value. For $\lambda > \lambda_0$ the hunted apex in the diagram equations depends on the value of λ. The solutions of the equations, namely the tensors F(λ), G(λ), Γ(λ), can, in principle, be obtained from their values at $\lambda = \lambda_0$ by averaging over all transfers of pulsations from the scale interval $(\lambda_0 k^{-1}, \lambda k^{-1})$. The tensors obtained by this averaging must satisfy the relation $\mu_\lambda = F(\lambda)\cdot G^2(\lambda)\Gamma^2(\lambda)k^3w(\lambda) \sim 1$ since the considerations which have led to the result $\mu \sim 1$ retain their validity. This is confirmed by a straightforward calculation, as in [11], for the case of $\lambda = \infty$.

Thus, statistical characteristics of the semi-Lagrangian velocity are determined within the framework of the universal problem of strong interaction. Moments and response functions for the Eulerian velocity can be obtained with the aid of the formulas (17) and (19) in which the functions G_n, $\tilde{G}_n$ are tensors of rank n.

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RECONSTRUCTION OF TURBULENCE SPECTRUM FROM TRANSIENT CHARACTERISTICS OF A SHADOW-INSTRUMENT SIGNAL

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With the investigation of turbulence using a shadow instrument with photoelectric recording, the statistical characteristics of the signal taken off from the instrument are used to obtain information on the statistics of the investigated medium [1, 2]. In situations where the investigated medium is moving perpendicular to the instrument axis (for example, with experiments in hydro- and aerodynamic tubes), it is convenient to use the transient characteristics of the signal. In the present article an investigation is made of the connection of the transient correlation function and the frequency spectrum of a shadow-instrument signal with the energy spectrum of the optical inhomogeneities in the medium; a method is given for reconstructing the spectrum of the inhomogeneities from the correlation function or the transient spectrum of the signal.

§1. Connection between the Correlation Function of the Signal and the Fourth Moment of the Light Field

The general scheme of the shadow instrument is given in Fig. 1. A coherent monochromatic light beam from the illuminator 1 passes through a layer of the investigated medium with thickness L, situated between the planes 2 and 3, and is reflected by the lens 4 on its focal plane 5. In the plane 5 (the shadow plane) there is a shadow diaphragm; the light passing through the shadow plane is collected by the lens 6 and sent to the photomultiplier 7. In what follows, by the "instrument signal" we shall understand the intensity of the light falling on the photomultiplier (and not the photomultiplier current).

We introduce the Cartesian coordinates x, y, z in such a way that the z axis will be directed along the axis of the light propagation; plane 2 corresponds to $z = 0$, plane 3 to $z = L$. Let $u(x, y, L, t) \equiv u(\mathbf{x}, t)$, $\mathbf{x} = (x, y)$ be the random distribution of the field at the plane 3 at the moment of time t. Then the instantaneous value of the signal of the instrument E(t) is [3]

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